Lecture 2:

Three most basic techniques:

(1) Integrating factor

(2) Separation of variables

(3) Analytic spectral (Fourier) method

(1) Integrating factor

(A) First order differential equation (involving first derivatives ONLY)

Consider: (✱) $\frac{dy}{dx} + P(x)\,y(x) = Q(x)$ (y is unknown function)

Let $M(x) = e^{\int_{s_0}^{x} P(s)\,ds}$ some constant. Then, it is easy to check:

$$\frac{d}{dx}\big(M(x)\,y(x)\big) = M(x)\left(\frac{dy}{dx} + P(x)\,y(x)\right)$$

Multiply both sides of (✱) by M(x):

$$M(x)\left(\frac{dy}{dx} + P(x)\,y(x)\right) = M(x)\,Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\big(M(x)\,y(x)\big) = M(x)\,Q(x)$$

Example 2: Consider:

$$f(x)\frac{dy}{dx} + g(x)\,y(x) = h(x), \quad 2 \le x < \infty \quad \text{with } y(2) = 1$$

Suppose $f(x) \approx (x^2 - 1)$; $g(x) \approx 2x$; $h(x) \approx x$.

Find an approximated guess of $y(x)$.

Solution: Consider: $\dfrac{dy}{dx} + \dfrac{2x}{x^2-1}\,y(x) = \dfrac{x}{x^2-1}$

Let $M(x) = e^{\int \frac{2x}{x^2-1}\,dx} = e^{\ln(x^2-1)} = x^2 - 1$

Then: $M(x)\left(\dfrac{dy}{dx} + \dfrac{2x}{x^2-1}\,y(x)\right) = M(x)\left(\dfrac{x}{x^2-1}\right)$

$$\Rightarrow \frac{d}{dx}\big(M(x)\,y(x)\big) = M(x)\,\frac{x}{x^2-1}$$

$$\therefore \int \frac{d}{dx}\left((x^2-1)\,y(x)\right) = \int \cancel{(x^2-1)}\left(\frac{x}{\cancel{x^2-1}}\right)$$

$$\Rightarrow \quad (x^2-1)\,y(x) = \frac{x^2}{2} + C$$

$$\Rightarrow \quad y(x) = \left(\frac{1}{2}x^2 + C\right)\Big/(x^2-1)$$

$$y(2) = 1 \;\Rightarrow\; 1 = (C+2)/3 \;\Rightarrow\; C = 1.$$

$\therefore\;$ $y(x) = \left(\frac{1}{2}x^2+1\right)\Big/(x^2-1)$ is an approximated guess of the solution.

(B) Second order differential equation (involving second derivatives)

Consider : $-c \dfrac{d^2u}{dx^2} + q\,u(x) = 0$ where $c>0$, $q>0$ are positive constants.

Let $M(x) = \dfrac{du}{dx}$ (integrating factor)

Then: $c \dfrac{d^2u}{dx^2} \underbrace{M(x)}_{=\frac{du}{dx}} = q\,u(x) \underbrace{M(x)}_{=\frac{du}{dx}}$

$$\Longleftrightarrow \frac{d}{dx}\left(c\left(\frac{du}{dx}\right)^2 \right) = \frac{d}{dx}\left(q\,(u(x))^2 \right)$$

A possible solution of the above is :

$$c\left(\frac{du}{dx}\right)^2 = q\,(u(x))^2$$

$$\therefore \quad \frac{du}{dx} = \pm\sqrt{\frac{q}{c}}\;u(x)$$

Using the integrating factor technique for 1st order differential eqt:

$$u(x) = K e^{\pm\sqrt{\frac{q}{c}}x} \quad \text{for some constant } K.$$

For general solution, $u(x) = \alpha_1 e^{\sqrt{\frac{q}{c}}x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}x}$

where α_1 and α_2 are some constants determined by boundary conditions.

<u>Example</u>: Assume $u(0) = 0$ and $u(1) = 2$.

We get

$$\alpha_1 + \alpha_2 = 0$$

$$\alpha_1 e^{\sqrt{\frac{q}{c}}} + \alpha_2 e^{-\sqrt{\frac{q}{c}}} = 2$$

$$\Rightarrow \alpha_1 = -\alpha_2 = \frac{2}{e^{\sqrt{\frac{q}{c}}} - e^{-\sqrt{\frac{q}{c}}}}$$

Example: (Non-homogeneous case)

Consider:

$$(*)\begin{cases} -c\dfrac{d^2u}{dx^2} + qu = qx^2+1 \\ \dfrac{du}{dx}(0) = 1 \ , \quad u(1) = 1 \end{cases}$$

Note that if $w(x)$ satisfies $(*)$, then:

$$u(x) = \underbrace{\alpha_1 e^{\sqrt{\frac{q}{c}}x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}x}}_{\text{Homogeneous}} + \underbrace{w(x)}$$ for some constants

α_1 and α_2 is a general sol.

In our case, $w(x) = x^2 + \left(\frac{2c+1}{q}\right)$ is a solution.

$$\therefore\ u(x) = \alpha_1 e^{\sqrt{\frac{q}{c}}x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}x} + x^2 + \left(\frac{2c+1}{q}\right)$$

determined by boundary conditions.

Example: Solve: $\begin{cases} -2\dfrac{d^2u}{dx^2} + 4u(x) = 4x^2 - 4x + 12 & \text{for } 0 < x < 1 \quad (*) \\ u(0) = 1 \text{ and } u(1) = 1 \end{cases}$

Step 1: Solve the homogeneous eqt first: $-2\dfrac{d^2u}{dx^2} + 4u(x) = 0$

- **Multiply both sides by $M(x) = \dfrac{du}{dx}$**

$$-2\frac{du}{dx}\frac{d^2u}{dx^2} + 4u(x)\frac{du}{dx} = 0 \iff \frac{d}{dx}\left(-2\left(\frac{du}{dx}\right)^2\right) + \frac{d}{dx}\left(4\left(u(x)\right)^2\right) = 0$$

- **Guess possible solutions:**

$$-2\left(\frac{du}{dx}\right)^2 + 4\left(u(x)\right)^2 = 0 \iff 2\left(\frac{du}{dx}\right)^2 = 4u(x)^2 \iff \frac{du}{dx} = \pm\sqrt{2}\,u(x) \text{ are possible solutions}$$

$$\iff u_1(x) = \alpha_1 e^{\sqrt{2}x} \text{ and } u_2(x) = \alpha_2 e^{-\sqrt{2}x} \text{ are possible solutions}$$

$\therefore u(x) = \alpha_1 e^{\sqrt{2}x} + \alpha_2 e^{-\sqrt{2}x}$ (for some α_1 and α_2) is a solution for $(*)$

Step 2: Guess a particular solution $w(x)$

Guess: $w(x) = a_2x^2 + a_1x + a_0$ and put it into $(*)$.

We get: $-2(2a_2) + 4(a_2x^2 + a_1x + a_0) = 4x^3 - 4x + 12 \iff 4a_2x^2 + 4a_1x + 4a_0 - 4a_2 = 4x^2 - 4x + 12$

$$\iff 4a_2 = 4\ ;\ 4a_1 = -4\ ;\ 4a_0 - 4a_2 = 12 \iff a_2 = 1,\ a_1 = -1,\ a_0 = 4$$

$\therefore w(x) = x^2 - x + 4$ is a particular solution.

Step 3: Construct general sols and substitute boundary conditions

General sols: $u(x) = \alpha_1 e^{\sqrt{2}x} + \alpha_2 e^{-\sqrt{2}x} + (x^2 - x + 4)$ is a general sol because:

$$\underbrace{\left[-2\,\frac{d^2}{dx^2}\left(\alpha_1 e^{\sqrt{2}x}+\alpha_2 e^{-\sqrt{2}x}\right)+4\left(\alpha_1 e^{\sqrt{2}x}+e^{-\sqrt{2}x}\right)\right]}_{\substack{\parallel \\ 0}}+\underbrace{\left[-2\,\frac{d^2}{dx^2}\left(x^2-x+4\right)+4\left(x^2-x+4\right)\right]}_{4x^3-4x+12}=4x^3-4x+12$$

Boundary conditions: $\left(u(x)=\alpha_1 e^{\sqrt{2}x}+\alpha_2 e^{-\sqrt{2}x}+4x^3-4x+12\right)$

$$u(0)=\alpha_1+\alpha_2+12=1$$

$$u(1)=\alpha_1 e^{\sqrt{2}}+\alpha_2 e^{-\sqrt{2}}+12=1$$

$$\Longleftrightarrow\begin{cases}\alpha_1+\alpha_2=-11\\ \alpha_1 e^{\sqrt{2}}+\alpha_2 e^{-\sqrt{2}}=-11\end{cases}\quad\text{(Linear system)}$$

$$\Downarrow$$

Determine α_1 and α_2 (Exercise)

Another useful technique: Separation of variables

Consider a heat equation (on a unit circle):

$$u_t = u_{xx}, \quad x \in [0, 2\pi], \quad t \geq 0$$

subject to: $\begin{cases} u(0,t) = u(2\pi, t) & \text{(periodic condition)} \\ u(x,0) = \sin x & \text{(initial condition)} \end{cases}$

Strategy: Let $u(x,t) = X(x)\,T(t)$.

$$u_t = u_{xx} \Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$$

$$\therefore \frac{X''}{X} = \frac{T'}{T} = \lambda \leftarrow \text{some constant}$$

$$\Rightarrow \frac{d^2 X(x)}{dx^2} = \lambda X(x) \quad \text{and} \quad \frac{d}{dt} T(t) = \lambda T(t)$$

(Two differential equations!)